Chapter II Conforming Finite Elements

Note Title 1/28/2009

historical remarks

math: Courant (1943), Schellbach (1851), also Euler

Babuska & Aziz (1972), Strang & Fix (1973)

engineering (structural mechanics)

Turner, Clough, Martin, Topp (1956) — 1st paper finite element

Argyris (1957)

Zienkiewicz (1971) book, Oden (1991) review

§1 Sobolev Spaces

Assume that $\Omega \subset R^n$ is open with piecewise smooth boundary $\partial\Omega$.

- $L^2(\Omega) = \left\{ v \,\middle|\, \int_\Omega v^2 dx < +\infty \right\}$

$$(u,v)_0 = (u,v)_{L^2(\Omega)} = \int_\Omega uv\,dx, \qquad \|u\|_0 = \sqrt{(u,u)_0} = \sqrt{\int_\Omega u^2 dx}$$

multiple index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \geq 0$ integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$

- weak derivative

$u \in L^2(\Omega)$ has the (weak) derivative $v = \partial^\alpha u \in L^2(\Omega)$

$$\Longleftrightarrow \quad (\phi, v)_0 = (-1)^{|\alpha|} (\partial^\alpha \phi, u)_0 \quad \forall\, \phi \in C_0^\infty(\Omega)$$

example

$\alpha=(1,0,\cdots,0)$ $\displaystyle\int_\Omega \phi \frac{\partial u}{\partial x_1}\,dx = (-1)\int_\Omega \frac{\partial \phi}{\partial x_1}\,u\,dx$

$\alpha=(1,1,0,\cdots,0)$ $\displaystyle\int_\Omega \phi \frac{\partial^2 u}{\partial x_1 \partial x_2}\,dx = (-1)\int_\Omega \frac{\partial \phi}{\partial x_2}\frac{\partial u}{\partial x_1}\,dx = (-1)^2\int_\Omega \frac{\partial^2 \phi}{\partial x_1 \partial x_2}\,u\,dx$

- $\vec{u} \in L^2(\Omega)^n$ has divergence $v = \operatorname{div} \vec{u} \in L^2(\Omega)$

$$\Leftrightarrow \quad (\phi, v)_0 = -(\nabla\phi, \vec{u}) \quad \forall \phi \in C_0^\infty(\Omega).$$

Sobolev Spaces

Given an integer $m \geq 0$

- $H^m(\Omega) = \{ u \in L^2(\Omega) \mid \partial^\alpha u \text{ exists } \forall |\alpha| \leq m \}$

$$(u,v)_m = \sum_{|\alpha|\leq m} (\partial^\alpha u, \partial^\alpha v)_0 , \quad \|u\|_m = \sqrt{(u,u)_m} , \quad |u|_m = \sqrt{\sum_{|\alpha|=m} \|\partial^\alpha u\|_0^2}$$

- $H^m(\Omega)$ is complete $\longrightarrow$ $H^m(\Omega)$ is a Hilbert space

  $H^m(\Omega) \cap C^\infty(\Omega)$ is dense in $H^m(\Omega)$

  $H_0^m(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_m} = \{ u \in H^m(\Omega) \mid \partial^\alpha u|_{\partial\Omega} = 0 \ \forall |\alpha| \leq m \}$

- $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ are based on the L^p-norm

Friedrich's Inequality

- Assume that $\Omega \subset W = \{ x = (x_1, \cdots, x_n) \mid 0 < x_i < s \}$

$$\Rightarrow \quad \|v\|_0 \leq s\,|v|_1 \quad \forall v \in H_0^1(\Omega)$$

Proof Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it suffices to prove

$$\|v\|_0 \leq s\,|v|_1 \quad \forall v \in C_0^\infty(\Omega)$$

extension of v onto W:

$$v = \begin{cases} v & \text{in } \Omega \\ 0 & \text{in } W \setminus \Omega \end{cases}$$

$$v(x) = \int_0^{x_1} \partial_1 v(t, x_2, \cdots, x_n)\,dt \leq s^{\frac{1}{2}} \left(\int_0^s |\partial_1 v|^2 dx_1 \right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \int_0^s |v|^2 dx_1 \leq s^2 \int_0^s |\partial_1 v|^2 dx_1$$

$$\Rightarrow \quad \|v\|_{0,\Omega}^2 = \|v\|_{0,W}^2 = \int_W |v|^2 dx \leq s^2 \int_W |\partial_1 v|^2 dx \leq s^2 \int_W |\nabla v|^2 dx = s^2 |v|_{1,\Omega}^2$$

Thrm Assume that Ω is bounded

$$\Rightarrow \quad |\cdot|_m \text{ and } \|\cdot\|_m \text{ are equivalent on } H_0^m(\Omega): \quad |v|_m \leq \|v\|_m \leq C_\Omega |v|_m \quad \forall v \in H_0^m(\Omega)$$

Remark Poincaré's Inequality holds on

$$H_D^1(\Omega) = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$$

$$\hat{H}^1(\Omega) = \{ v \in H^1(\Omega) \mid \int_\Omega v\,dx = 0 \}$$

Problem #1.12

Possible Singularities of H^1 functions

- $L^2(\Omega)$ contains unbounded functions for all n-dim

  $\underline{n=1}$ $\Omega = (0,1)$

  $$v(x) = x^{-\alpha} \quad \text{for } 0<\alpha<\tfrac{1}{2} \qquad \in H^1(0,1)$$

- $H^1(\Omega)$ contains unbounded functions for $n \geq 2$ dimensions.

  $\underline{n=1}$ $H^1(a,b) \subset C^0(a,b)$

  $\underline{n=2}$ $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$

  $$u(x,y) = \log\log\frac{2}{r} \in H^1(\Omega) \quad \text{since} \int_0^1 |\partial_r u|^2 r\,dr = 4\int_0^{\frac{1}{2}} \frac{1}{r\log^2 r}\,dr < +\infty$$

  $\underline{n \geq 3}$ $u = r^{-\alpha}$ for $0<\alpha<\frac{n-2}{2}$ $\in H^1(D)$

Compact Imbeddings

Given $m \geq 0$, assume that Ω is a Lipschitz domain and that Ω satisfies a cone condition.

$\Longrightarrow$ the imbedding $H^{m+1}(\Omega) \hookrightarrow H^m(\Omega)$ is compact

(A cont. linear mapping $L: U \to V$ is compact $\Longleftrightarrow$ $L(B)$ is relatively compact)

unit ball in U

HWs 1.10, 1.11, 1.12.

§2. Variational Formulation of Elliptic BVPs of 2^{nd}-Order

Diffusion and Reaction Eq

$$(P)\ \begin{cases} Lu \equiv -\operatorname{div}(A\nabla u) + a_0 u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \quad \left| \begin{array}{l} a_0 \geq 0 \quad \text{in } \Omega \\ \alpha \|\xi\|_{\ell_2}^2 \leq \langle A\xi, \xi\rangle \ \forall \xi \in \mathbb{R}^n, \text{ in } \Omega \end{array} \right.$$

classical solution if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ Dirichlet BCs

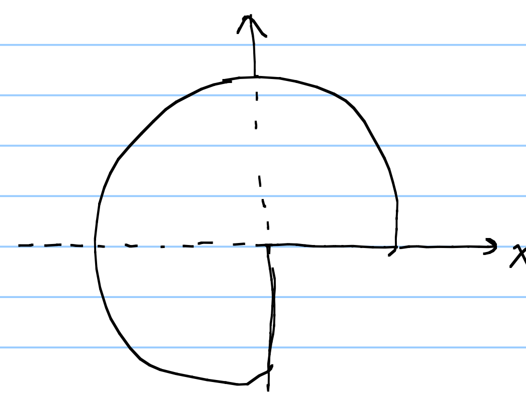
or $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ Neumann BCs

requirements $\partial\Omega$, given data — sufficiently smooth

additional conditions on mixed BCs

example (reentrant corner $\frac{3\pi}{2}$)

$$u(r,\theta) = r^{\frac{2}{3}} \sin\left(\frac{2\theta}{3}\right)$$



$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(1,\theta) = \sin\left(\frac{2\theta}{3}\right) & 0 < \theta < \frac{3\pi}{2} \\ u(r,\theta) = 0 & \theta = 0, \frac{3\pi}{2} \end{cases}$$

$$\Omega = \{(x,y) \mid x^2 + y^2 < 1,\ x < 0 \text{ or } y < 0\}$$

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$$

- $\partial_r u = \frac{2}{3} r^{-\frac{1}{3}} \sin\frac{2\theta}{3}$ unbounded as $r \to 0$ $\longrightarrow$ no classical der.
- but $u \in H^{1+\frac{2}{3}}(\Omega)$ $\longrightarrow$ weak der. exists

(M) $J(u) = \min_{v \in U} J(v)$ with $J(v) = \frac{1}{2} a(v,v) - \langle l, v \rangle$

(V) Find $u \in U$ s.t.

$$a(u,v) = \langle l, v \rangle \quad \forall v \in U.$$

Characterization Thrm

Let V be a linear space and $l : V \to \mathbb{R}$ be a linear funct.

Assume that $a : V \times V \to \mathbb{R}$ is sym. bilinear form.

$\Longrightarrow$ "$J(u) = \min_{v \in V} J(v)$ $\Longleftrightarrow$ Find $u \in V$ s.t. $a(u,v) = \langle l, v \rangle \ \forall v \in V$.

Moreover, $a(\cdot,\cdot)$ is pos. $\left(a(v,v) > 0 \ \forall 0 \neq v \in V \right)$

$\Longrightarrow$ (M) and (V) with $U = V$ have a unique solution.

Proof $\forall u, v \in V$, $t \in \mathbb{R}$

$$J(u+tv) = J(u) + t\left[a(u,v) - \langle l, v \rangle\right] + \frac{1}{2} t^2 a(v,v) \qquad (2.3)$$

$\Leftarrow$ • "$u \in V$ satisfies (2.2)"

(2.3) with $t=1$ $\Longrightarrow$ $\boxed{J(u+v) = J(u) + \frac{1}{2} a(v,v) > J(u) \ \forall 0 \neq v \in V}$

$\Longrightarrow$ u is a minimal pt.

$\Rightarrow$ • "$J(u) = \min_{v \in V} J(v)$"

Let $g(t) = J(u+tv)$ $\Longrightarrow$ $g(0) = \min_t g(t)$

$$\Longrightarrow g'(t)\Big|_{t=0} = 0 \Longrightarrow a(u,v) - \langle l, v\rangle = 0 \quad \forall v \in V.$$

- Uniqueness $\quad a(u_i, v) = \langle l, v\rangle \quad \forall v \in V$

$$\Longrightarrow a(u_1 - u_2, v) = 0 \Longrightarrow a(u_1-u_2, u_1-u_2) = 0 \Longrightarrow u_1 = u_2$$

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<u>Minimal Property</u>

Assume that a solution u of (2.6) is in V.

$$\Longrightarrow J(u) = \min_{v\in V_0} J(v) \qquad \text{with } J(v) = \frac{1}{2}\Big((A\nabla v, \nabla v) + (a_0 v, v)\Big) - (f, v)$$

<u>Proof</u> Let $a(u,v) = (A\nabla u, \nabla v) + (a_0 u, v)$ and $\langle l, v\rangle = (f, v)$

$\forall v \in V_0$

$$\Longrightarrow a(u,v) - \langle l, v\rangle = (A\nabla u, \nabla v) + (a_0 u, v) - (f, v)$$

$$= \int_\Omega (Lu - f)\, v\, dx = 0$$

clara. thrm

$$\Longrightarrow J(u) = \min_{v\in V_0} J(v)$$

#

$$\boxed{\text{Assume } \underline{\exists} u \text{ s.t. } J(u) = \min_{v\in V_0} J(v)} \Longrightarrow \begin{cases} Lu = f & \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

Existence of $\min J(v)$ in $C^2(\Omega) \cap C^0(\bar\Omega)$??

Existence of Solutions of Minimization Prob

Def. Let H be a Hilbert space, $a : H \times H \to \mathbb{R}$ be a bilinear form

a is continuous $\iff$ $\exists\, C > 0$ s.t. $|a(u,v)| \le C \|u\| \|v\| \quad \forall u, v \in H$

a is H-elliptic $\iff$ $\exists\, \alpha > 0$ s.t. $\underline{\alpha \|v\|^2 \le a(v,v) \quad \forall v \in H}$

$\Uparrow$

if dim H is finite, $a(v,v) > 0$

energy norm $\quad \|v\|_a = \sqrt{a(v,v)}$

The Lax-Milgram Thrm (Convex set)

Let V be a closed convex set in a Hilbert space H

let $a : H \times H \to \mathbb{R}$ be a cont. elliptic bilinear form

$\Longrightarrow \forall\, \ell \in H', \quad \exists\, u \in V$ s.t.

$$J(u) = \min_{v \in V} J(v) \qquad \text{with } J(v) = \frac{1}{2} a(v,v) - \langle \ell, v \rangle$$

Proof (Existence)

- J is bounded below

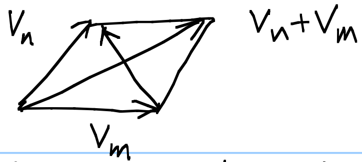
$$J(v) \geq \frac{1}{2} \alpha \|v\|^2 - \|\ell\|_{H'} \|v\| = \frac{\alpha}{2} \left(\|v\| - \frac{1}{\alpha} \|\ell\|_{H'} \right)^2 - \frac{\|\ell\|_{H'}^2}{2\alpha} \geq -\frac{\|\ell\|^2}{2\alpha}$$

- Let $c_1 = \inf_{v \in V} J(v)$ $\quad \left(\forall \varepsilon > 0, \ \exists\, v_\varepsilon \in V \text{ s.t. } J(v_\varepsilon) \le c_1 + \varepsilon \right)$

$$\Longrightarrow \exists \{v_n\}_{n=1}^\infty \text{ s.t. } \quad c_1 \le J(v_n) \le c_1 + \frac{1}{n}$$

minimizing sequence

- $\{v_n\}_{n=1}^\infty$ is a Cauchy sequence in H



$$\alpha\|v_n - v_m\|^2 \le a(v_n - v_m, v_n - v_m) \overset{\text{Parallelogram law}}{=} 2a(v_n,v_n) + 2a(v_m,v_m) - a(v_n+v_m, v_n+v_m)$$

$$\left(4\left[\tfrac{1}{2}a(v_n,v_n) - \langle l, v_n\rangle\right] + 4\left[\tfrac{1}{2}a(v_m,v_m) - \langle l, v_m\rangle\right] - \left[a(v_n+v_m, v_n+v_m) - 4\langle l, v_n+v_m\rangle\right]\right)$$

$$= 4J(v_n) + 4J(v_m) - 8J\left(\frac{v_n+v_m}{2}\right)$$

$$\left(V \text{ is convex} \Rightarrow \frac{v_n+v_m}{2} \in V \Rightarrow J\left(\frac{v_n+v_m}{2}\right) \ge c_1\right)$$

$$\le 4J(v_n) + 4J(v_m) - 8c_1$$

$$\to 4c_1 + 4c_1 - 8c_1 = 0 \quad \text{as } n,m \to \infty$$

- $\lim\limits_{n\to\infty} v_n = u \overset{V \text{ is closed}}{\Longrightarrow} u \in V$

- $J(u) = \lim\limits_{n\to\infty} J(v_n) = \inf\limits_{v\in V} J(v)$.

<u>(Uniqueness)</u> Assume that it has 2 solutions u_1 and u_2

$\Rightarrow \{u_1, u_2, u_1, u_2, \cdots\}$ is a minimizing sequence

$\Rightarrow$ it is a Cauchy sequence $\Rightarrow \|u_1 - u_2\| = 0 \Rightarrow u_1 = u_2$. #

<u>Remark</u> (1) $a(\cdot,\cdot)$ defines an inner product in H

<u>Riesz Rep. Thrm</u> $\Rightarrow \exists\, \sigma l \in H$ s.t. $\langle l, v\rangle = a(\sigma l, v) \quad \forall v \in H$

$$\Rightarrow J(v) = \tfrac{1}{2}a(v,v) - a(\sigma l, v) = \tfrac{1}{2}a(v-\sigma l, v-\sigma l) - \tfrac{1}{2}a(\sigma l, \sigma l)$$

$$\Rightarrow \text{"}J(u) = \min_{v\in V} J(v) \Longleftrightarrow \|u - \sigma l\|_a = \min_{v\in V}\|v - \sigma l\|_a\text{"}$$

u is the projection of σl onto V.

(2) $V = H$ Given $\ell \in H'$, $\exists\, u \in H$ s.t. $a(u,v) = \langle \ell, v\rangle \quad \forall v \in H$ $\Big\} \iff u = \sigma\ell$

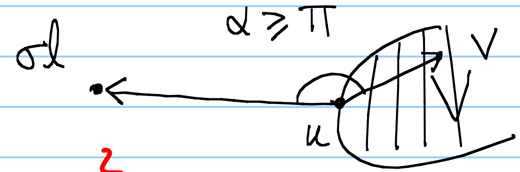
(3) V is a closed convex set.

#2.13

$$J(u) = \min_{v \in V} J(v) \iff u \in V, \quad a(u, v-u) \geq \langle \ell, v-u\rangle \quad \forall v \in V.$$

Proof $J(u) = \min_{v\in V} J(v) \iff \|u - \sigma\ell\|_a = \min_{v \in V} \|v - \sigma\ell\|_a$

$$\overset{?}{\iff} \quad a(\sigma\ell - u, v-u) \leq 0 \quad \forall v \in V \iff \langle \ell, v-u\rangle \leq a(u, v-u) \ \forall v \in V$$



$$\|u - \sigma\ell\|_a^2 \leq \|v - \sigma\ell\|_a^2 \quad \forall v \in V$$

$$\iff 0 \geq \|u-\sigma\ell\|_a^2 - \|v - \sigma\ell\|_a^2$$

$$= \|u - \sigma\ell\|_a^2 - \|(v-u) + (u - \sigma\ell)\|_a^2$$

$$= -\|v-u\|_a^2 + 2a(\sigma\ell - u, v-u)$$

$$\iff \|v-u\|_a^2 \geq a(\sigma\ell - u, v-u) \quad \forall v \in V$$

$$\overset{?}{\iff} \quad 0 \geq a(\sigma\ell - u, v-u) \quad \forall v \in V$$

2.8 Def. $u \in H_0^1(\Omega)$ is a weak solution of $\begin{cases} Lu = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$

if $u \in H_0^1(\Omega)$ satisfies $a(u,v) = (f,v)$ $\forall v \in H_0^1(\Omega)$.

2.9 Existence Thrm $a_{ij}, a_0 \in L^\infty(\Omega)$, $f \in L^2(\Omega)$

Variational problem: find $u \in H_0^1(\Omega)$ s.t. $a(u,v) = (f,v)$ $\forall v \in H_0^1(\Omega)$

always has a solution.

Moreover, $J(u) = \min_{v \in H_0^1(\Omega)} J(v)$.

Proof $a(u,v) = (A\nabla u, \nabla v) + (a_0 u, v)$

$$|a(u,v)| \leq c \|u\|_1 \|v\|_1 \quad \forall u,v \in H_0^1(\Omega)$$

$$a(v,v) \geq \alpha |v|_1^2 \quad \forall v \in H_0^1(\Omega)$$
$$\geq \alpha C_\Omega \|v\|_0^2$$

$$|\langle l, v\rangle| = |(f,v)| \leq \|f\|_0 \|v\|_0$$

#

example $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$

inhomogeneous BCs $\begin{cases} Lu = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

u_0 — extension of g into Ω $\in H^1(\Omega)$ s.t. $\begin{cases} Lu_0 \text{ exists} \\ u_0|_{\partial\Omega} = g \end{cases} \Longrightarrow \begin{cases} L(u-u_0) = f - Lu_0 & \text{in } \Omega \\ (u-u_0)|_{\partial\Omega} = 0 \end{cases}$

Find $w = u - u_0 \in H_0^1(\Omega)$ s.t.

$$a(w, v) = (f - Lu_0, v)_0 \quad \forall v \in H_0^1(\Omega)$$

$$\underset{a(u-u_0, v)}{\parallel} \qquad \underset{(f,v) - a(u_0, v)}{\parallel}$$

$\Longrightarrow$ Find $u \in H^1(\Omega)$ s.t.

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

$$u - u_0 \in H_0^1(\Omega).$$

<u>Summary</u>

(P) $\begin{cases} Lu \equiv -\operatorname{div}(A\nabla u) + a_0 u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$

(1) <u>Chara. Thrm</u> V — linear space, $l : V \to R$ linear, $a : V \times V \to R$ sym. bilinear.

(M) $J(u) = \min_{v \in V} J(v) \iff \begin{bmatrix} \text{Find } u \in V \\ a(u,v) = \langle l, v \rangle \ \forall v \in V \end{bmatrix}$ (V)

(2) $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a solution of (P) $\xrightarrow{\text{Minimal Principle}}$ u is a solution of (M)

u ———— $\Longleftarrow$ $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is ————

it is possible that there is NO $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ s.t. $J(u) = \min_{v \in C^2(\Omega) \cap C^0(\bar{\Omega})} J(v)$

(3) (Existence) The Lax-Milgram Thrm (closed convex set)

H — Hilbert space, $V \subset H$ — closed convex set, $a : H \times H \to R$ cont. elliptic bilinear

$\Longrightarrow \forall l \in H', \exists u \in V$ s.t. $J(u) = \min_{v \in V} J(v)$.

$\Updownarrow$ $\forall l \in H', \exists u \in V$ s.t. $a(u, v-u) \geq \langle l, v-u \rangle \ \forall v \in V.$

(4) (V) for (P): Find $u \in H_0^1(\Omega)$ s.t. $a(u,v) = \langle l, v \rangle \ \forall v \in H_0^1(\Omega)$

(5) Inhomogeneous BCs.

§3. The Mixed BVPs. A Trace Thrm

(P) $$\begin{cases} -\operatorname{div}(A\nabla u) + a_0 u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ A\nabla u\cdot\vec{n} = g & \text{on } \Gamma_N \end{cases}$$

$$\overline{\partial\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N}$$

$$(f,v) = (-\operatorname{div}(A\nabla u) + a_0 u,\ v) \nearrow -\int_{\Gamma_N} g v\, ds \quad \text{if } v|_{\Gamma_N} = 0$$

$$= (A\nabla u, \nabla v) - \int_{\partial\Omega} (\vec{n}\cdot A\nabla u)\, v\, ds + (a_0 u, v)$$

(V) Find $u \in H^1_D(\Omega) = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$ s.t.

$$a(u,v) = f(v) \quad \forall\, v \in H^1_D(\Omega)$$

$\Updownarrow$ where $a(u,v) = (A\nabla u, \nabla v) + (a_0 u, v)$

$$f(v) = (f,v) + \int_{\Gamma_N} g v\, ds$$

(M) Find $u \in H^1_D(\Omega)$ s.t.

$$J(u) = \min_{v\in H^1_D(\Omega)} J(v) \quad \text{with } J(v) = \frac{1}{2}a(v,v) - f(v)$$

$$f \in L^2(\Omega)$$

Thrm $(P) \Rightarrow (V)$ and $(V) + u \in C^2(\Omega)\cap C^1(\bar\Omega) \Rightarrow (P)$

Proof $(V) \xRightarrow{u\in C^2} (P)$ $\quad u \in H^1_D(\Omega) \Rightarrow u|_{\Gamma_D} = 0$

$$0 = f(v) - a(u,v) = \int_\Omega \left[f + \operatorname{div}(A\nabla u) - a_0 u\right] v\, dx + \int_{\Gamma_N}\left[g - \vec{n}\cdot A\nabla u\right] v\, ds - \int_{\Gamma_D} (\vec{n}\cdot A\nabla u)\, v\, ds$$

$$v \in C_0^\infty(\Omega) \Rightarrow 0 = \int_\Omega \left[f + \operatorname{div}(A\nabla u) - a_0 u\right] v\, dx = 0 \Rightarrow -\operatorname{div}(A\nabla u) + a_0 u = f \text{ in } \Omega$$

$$v \in H^1_D(\Omega) \Rightarrow 0 = \int_{\Gamma_N} \left[g - \vec{n}\cdot A\nabla u\right] v\, ds \xRightarrow[\text{chosen arbitrarily}]{v|_{\Gamma_N} \text{ can be}} \vec{n}\cdot A\nabla u\big|_{\Gamma_N} = g$$

#

Thrm (V) has a unique solution if $\text{mes}(\Gamma_D) \neq 0$. $(a_0 \geq 0)$

Proof $a(\cdot,\cdot)$ is sym, cont., elliptic $\quad \|v\|_{0,\Gamma_N} \leq C\|v\|_{1,\Omega}$

$$|f(v)| \leq \|f\|\,\|v\| + \|g\|_{0,\Gamma_N}\|v\|_{0,\Gamma_N} \leq C\left(\|f\| + \|g\|_{0,\Gamma_N}\right)\|v\|_{1,\Omega}.$$

$$\leq \|f\|_{-1}\|v\|_1 + \|g\|_{-\frac{1}{2},\Gamma_N}\|v\|_{\frac{1}{2},\Gamma_N}$$

$\|v\|_{\frac{1}{2},\Gamma_N} \leq C\|v\|_{1,\Omega}$

$$\leq C\left(\|f\|_{-1} + \|g\|_{-\frac{1}{2},\Gamma_N}\right)\|v\|_{1,\Omega}$$

#

Pure Neumann BCs $(\Gamma_D = \emptyset)$ and $a_0 = 0$

solution is unique up to an additive const.

$$\hat{H}^1(\Omega) = \left\{ v \in H^1(\Omega) \,\middle|\, \int_\Omega v\,dx = 0 \right\}$$

$$a(v,v) = (A\nabla v, \nabla v) \geq \alpha\|\nabla v\|^2 \geq C\|v\|_1^2.$$

the compatibility condition $(A = I,\ a_0 = 0,\ \Gamma_N = \partial\Omega)$

$$\int_\Omega f\,dx = \int_\Omega -\Delta u\,dx = -\int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds = -\int_{\partial\Omega} g\,ds$$

$$\Longrightarrow \int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0$$

A Trace Thrm Suppose that Ω has a Lipschitz boundary.

$\forall\, p \in [1,+\infty]$, $\exists\, C>0$ s.t. $\boxed{u \in W^{1,p}(\Omega) \Longrightarrow u|_{\partial\Omega} \in L^p(\partial\Omega)}$

$$\|u\|_{p,\partial\Omega} \leq C\, \|u\|_{L^p(\Omega)}^{1-\frac{1}{p}}\ \|u\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \qquad \forall u \in W^{1,p}(\Omega).$$

Proof special case $p=2$ and $\Omega = \{(x,y)\in \mathbb{R}^2 \mid x^2+y^2<1\}$

$$= \{(r,\theta) \mid 0 \leq r<1,\ 0\leq \theta < 2\pi\}$$

(1) $u \in C^1(\bar\Omega)$

$$\|\ \|_{0,\partial\Omega} \leq 2^{\frac{3}{4}}\, \|\ \|_{0,\Omega}^{\frac{1}{2}}\ \|\ \|_{1,\Omega}^{\frac{1}{2}}.$$

Proof $u^2(1,\theta) \overset{?}{=} \int_0^1 \frac{\partial}{\partial r}\left(r^2u^2\right)dr$ $\qquad \frac{\partial u}{\partial r} = \nabla u \cdot \frac{(x,y)}{r}$

$$= 2\int_0^1 \left(r^2 u\, \nabla u\cdot \frac{(x,y)}{r} + r u^2\right) dr$$

$$\leq 2\int_0^1 \left(|u|\,|\nabla u| + u^2\right) r\,dr$$

$$\Longrightarrow \|u\|_{0,\partial\Omega}^2 = \int_0^{2\pi} u^2(1,\theta)\, d\theta \leq 2\int_\Omega \left(|u|\,|\nabla u| + u^2\right)dxdy \leq 2^{\frac{3}{2}} \|u\|_{0,\Omega}\, \|u\|_{1,\Omega}.$$

(2) $u \in H^1(\Omega)$

$C^1(\bar\Omega)$ is dense in $H^1(\Omega) \Longrightarrow \exists\, \{u_j\} \subset C^1(\bar\Omega)$ s.t. $\|u-u_j\|_{1,\Omega} \leq \frac{1}{j}$

$$\Longrightarrow \|u_k - u_j\|_{0,\partial\Omega} \leq C\, \|u_k - u_j\|_{1,\Omega} \to 0$$

$\Longrightarrow \{u_j\}$ is a Cauchy sequence in $L^2(\partial\Omega)$

$\xrightarrow{L^2(\partial\Omega) \text{ is complete}}$ $\exists\, v \in L^2(\partial\Omega)$ s.t. $\lim_{j} \|u_j - v\|_{0,\partial\Omega} = 0$

Define $u|_{\partial\Omega} = v$.

- $u|_{\partial\Omega}$ is indep. of $\{u_j\}$

<u>Proof</u> Assume that $\{v_j\} \subset C^1(\bar{\Omega})$ s.t. $\lim_j \|u - v_j\|_{1,\Omega} = 0$

$$\Longrightarrow \lim_j \|v - v_j\|_{0,\partial\Omega} = 0 \quad \left(\begin{array}{l} \|v - v_j\|_{0,\partial\Omega} \le \|v - u_j\|_{0,\partial\Omega} + \|u_j - v_j\|_{0,\partial\Omega} \\ \qquad\qquad\qquad\qquad\qquad\qquad \le c\,\|u_j - v_j\|_{1,\Omega} \end{array} \right)$$

- $\|u\|_{0,\partial\Omega} \leq c\,\|u\|_{0,\Omega}^{\frac{1}{2}} \|u\|_{1,\Omega}^{\frac{1}{2}}$

<u>Proof</u> $\|u\|_{0,\partial\Omega} = \|v\|_{0,\partial\Omega} = \lim_j \|u_j\|_{0,\partial\Omega}$

$$\le c \lim_j \|u_j\|_{0,\Omega}^{\frac{1}{2}} \|u_j\|_{1,\Omega}^{\frac{1}{2}} = c\,\|u\|_{0,\Omega}^{\frac{1}{2}} \|u\|_{1,\Omega}^{\frac{1}{2}}. \qquad \#$$

<u>HW</u> 3.6, 3.8

§4. The Ritz-Galerkin Method and Some Finite Element

(M) $J(u) = \min_{v\in V} J(v)$ with $J(v) = \frac{1}{2}a(v,v) - \langle l, v\rangle$

$V = H^1_D(\Omega)$ or $\hat{H}^1(\Omega)$

(V) Find $u \in V$ s.t. $a(u,v) = \langle l, v\rangle \ \forall v \in V$

finite dimensional subspace $S_h \subset V$

$$J(u_h) = \min_{v\in S_h} J(v) \iff \text{Find } u_h \in S_h \text{ s.t. } a(u_h, v) = \langle l, v\rangle \ \forall v \in S_h.$$

Let $S_h = \text{span}\{\psi_i(x)\}_{i=1}^N$

Find $u_h = \sum_j z_j \psi_j(x)$ s.t. $a(u_h, \psi_i) = \langle l, \psi_i\rangle \ \forall i$

$$Az = b \quad \text{with } A = \left(a(\psi_j, \psi_i)\right)_{N\times N} \quad b = \left(\langle l, \psi_i\rangle\right)_{N\times 1}$$

$\Rightarrow$ A is sym. pos. def.

Methods

- Rayleigh-Ritz method $\forall v \in S_h$, $v = \sum v_i \psi_i(x)$

$$J(u_h) = \min_{v_i} J\left(\sum v_i \psi_i(x)\right) \iff \frac{\partial J\left(\sum v_i \psi_i(x)\right)}{\partial v_j} = 0 \quad \forall j$$

- Galerkin method ($a(\cdot,\cdot)$ is not sym) Ritz-Galerkin — a is sym.

Find $u_h \in S_h$ s.t. $a(u_h, v) = \langle l, v\rangle \ \forall v \in S_h$

• **Petrov-Galerkin** $\quad \dim S_h = \dim T_h$

Find $u_h \in S_h$ s.t. $a(u_h, v) = \langle \ell, v\rangle \ \forall v \in T_h$

examples: finite volume element, singular function methods

Assumptions $\quad V \subset H^1(\Omega)$

(1) $a(\cdot,\cdot)$ is cont.: $|a(u,v)| \le C \|u\|_1 \|v\|_1 \quad \forall u,v \in V$

(2) $a(\cdot,\cdot)$ is V-elliptic: $\exists \alpha > 0$ s.t. $a(v,v) \ge \alpha \|v\|_1^2 \quad \forall v \in V$

(3) $\ell \in V'$: $|\langle \ell, v\rangle| \le \|\ell\|_* \|v\|_1 \quad \forall v \in V.$

Stability $\quad u_h$ — solution

$$\|u_h\|_1 \le \frac{1}{\alpha} \|\ell\|_*$$

Proof $\quad \alpha \|u_h\|_1^2 \le a(u_h, u_h) = \langle \ell, u_h\rangle \le \|\ell\|_* \|u_h\|_1$ #

Cea's Lemma

$$\|u - u_h\|_1 \le \frac{C}{\alpha} \inf_{v \in S_h} \|u - v\|_1$$

$a(u - u_h, v) = 0 \ \forall v \in S_h.$

Proof $\quad \alpha \|u - u_h\|_1^2 \le a(u - u_h, u - u_h) = a(u - u_h, u - v) \quad \forall v \in S_h$

$\le C \|u - u_h\|_1 \|u - v\|_1$

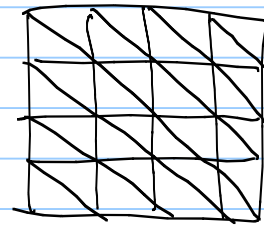
#

Projection Assume that $a(\cdot,\cdot)$ is sym. Let $\|v\|_a = \sqrt{a(v,v)}$.

$$\|u - u_h\|_a = \min_{v \in S_h} \|u - v\|_a.$$

Example (Courant 1943)

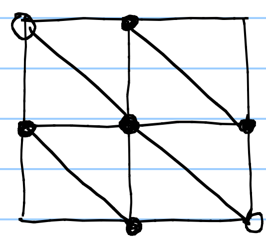
$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



$\mathcal{T}_h$: a uniform triangulation

$$S_h = \left\{ v \in C^0(\bar{\Omega}) \,\middle|\, v|_K \text{ is linear and } v|_{\partial\Omega} = 0 \right\}$$

$$= \operatorname{span}\{\psi_i(x)\} \quad \text{nodal basis functions}$$



$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

WH 4.4, 4.5, 4.8, 4.9

§5 Finite Element Spaces (Z. Cai)

$$V = H^1(\Omega) \text{ or } H^1_D(\Omega)$$

- Construction of $S^h \subset V$ where S^h is piecewise smooth functions
  (1) Partition of Ω
  (2) Global smoothness of p. smooth functions in V
  (3) Construction of local spaces
- Approximation Properties

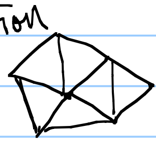
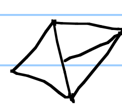
Partition of Ω

Def (Subdivision) $\{T_i\}$ is a subdivision of a domain Ω

$\Longleftrightarrow$ (1) K_i are open sets;
(2) $K_i \cap K_j = \phi$ if $i \neq j$;
(3) $\cup \overline{K}_i = \overline{\Omega}$

Def (Triangulation) $\mathcal{T} = \{K\}$ is a triangulation of a polygonal domain Ω

$\Longleftrightarrow$ (1) $\mathcal{T}$ is a subdivision of Ω;
(2) $\forall K \in \mathcal{T}$, K is a triangle (or rectangle)
(3) no vertex of any $K \in \mathcal{T}$ lies in the interior of an edge of $K' \in \mathcal{T}$

Triangulation   not

Global smoothness

<u>Thrm</u> Assume that Ω is bounded. Let $k \geq 1$ and v be a piecewise infinitely diff. function $v: \overline{\Omega} \to R$.

$\Longrightarrow$ " $v \in H^k(\Omega) \iff v \in C^{k-1}(\Omega)$ ".

<u>Proof ($k=1$)</u> Let v be p. infinitely diff w.r.t $\mathcal{T} = \{K\}$.

$\forall \varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (\partial_1 v)\varphi\, dx = \sum_K \int_K (\partial_1 v)\varphi\, dx = -\int_\Omega v\, \partial_1 \varphi\, dx + \sum_K \int_{\partial K} \varphi v n_1\, ds$$

$$v \in H^1(\Omega) \iff 0 = \sum_K \int_{\partial K} \varphi v n_i\, ds = \sum_F \int_F \varphi [\![v]\!]_F n_i\, ds \qquad F = \partial K \cap \partial K'$$

$$\iff [\![v]\!]_F = 0 \ \forall F \iff v \in C^0(\overline{\Omega})$$

#

Construction of Local FE Space

$(K, \mathcal{P}, \mathcal{N})$ is called a finite element

K — element, $\mathcal{P}$ — finite-dim space of functions on K

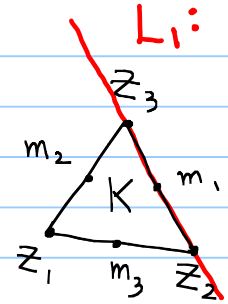
$\mathcal{N}$ — degrees of freedom

Triangular Finite Elements

K — triangle, $\mathcal{P}_k = \{$all poly. of degree $\leq k\}$, $\dim \mathcal{P}_k = \frac{1}{2}(k+1)(k+2)$

$\mathcal{N}_k = ?$

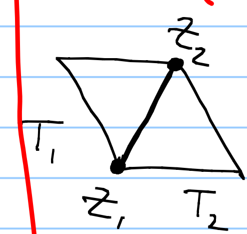
$\underline{k=1}$ $\quad L_1: \; L_1(x,y) = a_1 + b_1 x + c_1 y = 0$



$\mathcal{P}_1 = \text{span}\{1, x, y\}, \quad \mathcal{N}_1 = \{N_1, N_2, N_3\}$

- **Linear Lagrange Element** $\quad \underline{N_i(v) = v(z_i)}$

$\boxed{\text{in } C^0(\bar{\Omega})}$



$v_1|_e = v_2|_e$

Lemma ($\mathcal{N}_1$ determines $\mathcal{P}_1$)

$\forall v \in \mathcal{P}_1, \quad N_i(v) = v(z_i) = 0 \implies v \equiv 0 \text{ on } K$

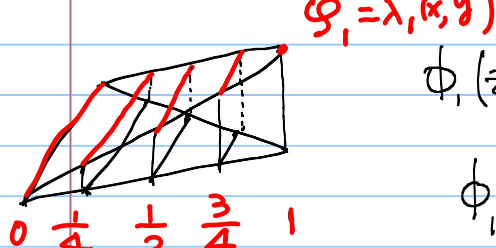
Proof $\quad v|_{L_1} = a + bs$ is linear and $v(z_2) = v(z_3) = 0$

$\implies v \equiv 0$ on $L_1 \implies v = c\, L_1(x,y)$ where c is a const.

$\xRightarrow{v(z_1)=0} \quad c = 0 \implies v \equiv 0$ on K. $\qquad$ #

Nodal Basis Functions $\quad N_j(\phi_i) = \phi_i(z_j) = \delta_{ij}$

$\varphi_1 = \lambda_1(x,y)$



$\phi_1(z_2) = \phi_1(z_3) = 0 \implies \phi_1|_{L_1} \equiv 0 \implies \phi_1 = c\, L_1(x,y)$

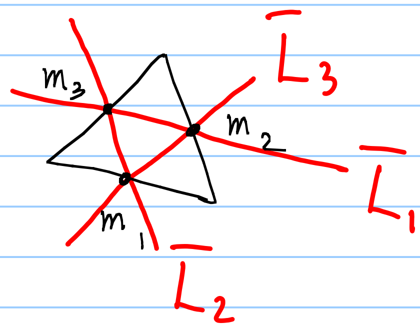
$\phi_1(z_1) = 1 \implies \phi_1(x,y) = \dfrac{L_1(x,y)}{L_1(z_1)} = \lambda_1(x,y)$

Barycentric coordinate $\quad \lambda_i(z_j) = \delta_{ij}$

$$\lambda_i(x,y) = \frac{L_i(x,y)}{L_i(z_i)}$$

$$\lambda_1(x,y) + \lambda_2(x,y) + \lambda_3(x,y) \equiv 1 \quad \text{on } K.$$

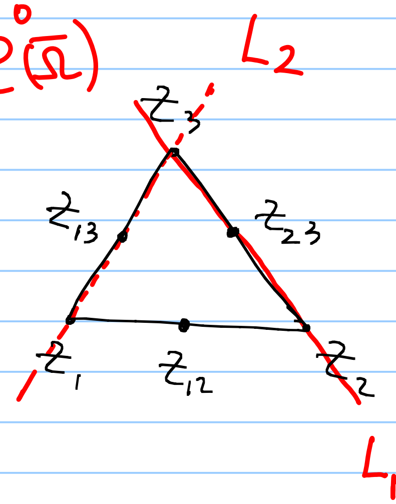
- Crouzeix–Raviart element $\quad \underline{N_i(v) = v(m_i)}$ not in $\overset{\circ}{C}(\bar{\Omega})$



(1) $\mathcal{N}_1$ determines $\mathcal{P}_1$

(2) Nodal basis functions $\quad \phi_i = \dfrac{\bar{L}_i(x,y)}{\bar{L}_i(m_i)}$

$\underline{k=2}$ (quadratic Lagrange element) in $\overset{\circ}{C}(\bar{\Omega})$



$$\mathcal{P}_2 = \text{span}\{1, x, y, x^2, xy, y^2\}$$

$$\mathcal{N}_2 = \{N_1, N_2, N_3, N_{12}, N_{13}, N_{23}\}$$

$$N_i(v) = v(z_i), \quad N_{ij}(v) = v(z_{ij})$$

Lemma ($\mathcal{N}_2$ determines $\mathcal{P}_2$)

$$\forall v \in \mathcal{P}_2, \quad v(z_i) = 0, \; v(z_{ij}) = 0 \implies v \equiv 0 \text{ on } K$$

Proof $\quad v\big|_{L_1} = a + bs + cs^2$ and $v(z_2) = v(z_{23}) = v(z_3) = 0$

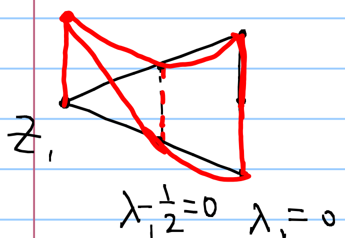
$$\implies v\big|_{L_1} \equiv 0 \implies v = L_1(x,y)\, Q_1(x,y) \quad \text{with } Q_1 \in \mathcal{P}_1$$

$$v\big|_{L_2} = L_1(x,y)\, Q_1(x,y)\Big|_{L_2} \equiv 0 \implies Q_1(x,y)\Big|_{L_2} = 0 \quad \text{except possible at } z_3$$

$$\implies Q_1\big|_{L_2} \equiv 0 \implies Q_1(x,y) = c\, L_2(x,y)$$

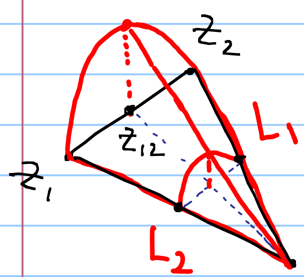
$$\implies v = c\, L_1(x,y)\, L_2(x,y) \xrightarrow{v(z_{12})=0} c = 0 \implies v \equiv 0. \qquad \#$$

Nodal Basis Functions



$$\left.\begin{aligned} \phi_1(x,y) &= c\,\lambda_1(x,y)\left(2\lambda_1 - 1\right) \\ 1 = \phi_1(z_1) &= c \end{aligned}\right\} \phi_1 = \lambda_1(2\lambda_1 - 1)$$

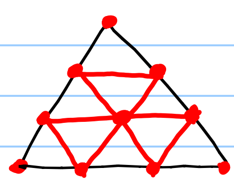
$$\phi_i(x,y) = \lambda_i(x,y)\left(2\lambda_i(x,y) - 1\right)$$



$$\left.\begin{aligned} \phi_{12} &= c\,\lambda_1\lambda_2 \\ 1 = \phi_{12}(z_{12}) &= c\,\lambda_1(z_{12})\,\lambda_2(z_{12}) = \frac{c}{4} \end{aligned}\right\}$$

$$\Longrightarrow \phi_{12} = 4\lambda_1\lambda_2, \quad \phi_{13} = 4\lambda_1\lambda_3, \quad \phi_{23} = 4\lambda_2\lambda_3.$$

$k = 3$ (Cubic Lagrange Element)



$\mathcal{P} = \mathcal{P}_3$, $\dim \mathcal{P}_3 = 10$

3		3 vertex nodes
$3(k-1)$	$3(3-1) = 6$	distinct edge nodes
$+)\ \frac{1}{2}(k-2)(k-1)$	$\frac{1}{2}(3-2)(3-1) = 1$	" interior nodes

$\dim \mathcal{P}_k = \frac{1}{2}(k+2)(k+1)$

$\mathcal{N}_3$ — nodal values

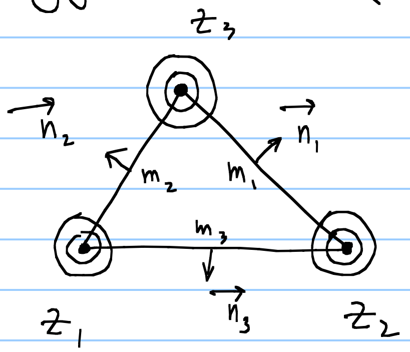
$\mathcal{N}_k$ — nodal values

$\mathcal{N}_3$ determines $\mathcal{P}_3$: $\forall v \in \mathcal{P}_3$ and $\mathcal{N}_3(v) = 0 \Rightarrow v = c\,L_1L_2L_3 \xrightarrow{v(\text{int. node}) = 0} c = 0 \Rightarrow v \equiv 0$

$\mathcal{N}_k$ determines $\mathcal{P}_k$: $v = L_1L_2L_3\,Q$ with $Q \in \mathcal{P}_{k-3}$

$\Longrightarrow Q = 0$ at all interior nodes $\Longrightarrow Q \equiv 0 \Longrightarrow v \equiv 0.$

The Argyris Element (C^1– element)

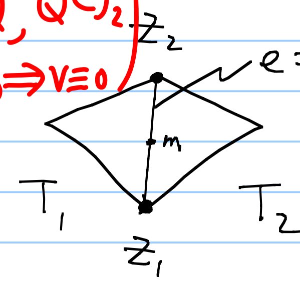


$$\mathcal{N} = \{N_i\}_{i=1}^{21}$$

•	values	3×1
○	gradient	3×2
◯	2^{nd} der.	3×3
•→	normal der.	3×1
		$3\times 7 = 21$

$\mathcal{P} = \mathcal{P}_5$, $\dim \mathcal{P}_5 = 21$

- $\mathcal{N}$ determines $\mathcal{P}_5$ $\left(\begin{array}{l} v|_{L_i} \equiv 0 \Rightarrow v = L_1 L_2 L_3 Q,\ Q \in \mathcal{P}_2 \\ Q(z_i) = 0,\ Q(m_i) = 0 \Rightarrow Q \equiv 0 \Rightarrow v \equiv 0 \end{array} \right)$
- The Argyris Element is in $C^1(\Omega)$



Proof $\mathcal{I}_i f \in \mathcal{P}_5(T_i)$ — interpolant:

$$N_j^i(\mathcal{I}_i f) = N_j^i(f), \quad i = 1, 2;\ j = 1, \ldots, 21$$

Set $w = \mathcal{I}_1 f - \mathcal{I}_2 f$

$w|_e = p_5(s)$ $\left\{ \begin{array}{l} 2 \text{ values} \\ 2\ 1^{st} \text{ der. along } s \\ 2\ 2^{nd} \text{ der.} \end{array} \right.$ $\underline{\hspace{1cm}}$ 6

On e: $w|_e = 0$ for 6 DoF $\Rightarrow w|_e \equiv 0 \Rightarrow C^0$–element

$$w|_e \equiv 0 \Rightarrow \boxed{\left.\frac{\partial w}{\partial \vec{t}}\right|_e = 0}$$

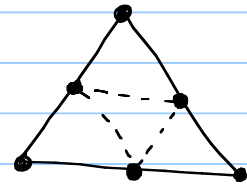
Let $r = \left.\frac{\partial w}{\partial n}\right|_e \in \mathcal{P}_4(e)$ and $r(z_1) = r(z_2) = r(m) = 0$

$$r'(z_1) = r'(z_2) = 0$$

$$\Rightarrow r \equiv 0 \Rightarrow \boxed{\left.\frac{\partial w}{\partial \vec{n}}\right|_e = 0} \Longrightarrow C^1\text{–element.}$$

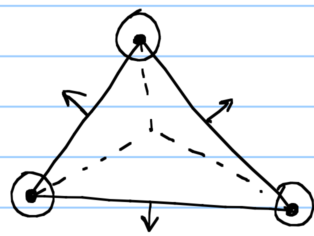
Composite Elements

- C^0 macro – piecewise linear triangle



- Hsieh–Clough–Tocher Element

T



dim $= 12$

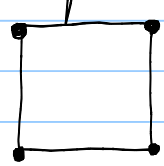
$u \in C^1(\Omega)$

$T = \bigcup_{i=1}^{3} K_i$

$u|_{K_i} \in \mathcal{P}_3$

Rectangular Elements

- bilinear quadratic element



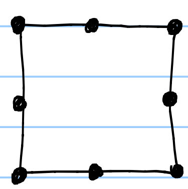
$u|_K \in \text{span}\{1, x, y, xy\} = Q_1$

$u \in C^0(\Omega)$

- Serendipity Element

$Q_2 = \text{span}\{1, x, y, xy, x^2, y^2, x^2y, xy^2, \underline{x^2y^2}\}$

$u \in C^0(\Omega)$



$u|_K \in \text{span}\{1, x, y, x^2, y^2, xy, x^2y, xy^2\}$

- **Approximation Properties**

$$S_h = \{ v \in C^0(\bar{\Omega}) \mid v|_K \in \mathcal{P}_k(K) \ \forall K \in \mathcal{T}_h \}$$

local interpolation operator

$$I_K v = \sum_i v(a_i) \varphi_i(x)$$

global interpolation operator

$$I_h v|_K = I_K v \quad \forall K \in \mathcal{T}_h \quad \forall v \in C^0(\bar{\Omega}).$$

affine-mapping $F_K : \hat{K} \to K$ by $\quad \hat{K}$ — a reference triangle with $\hat{h} = \text{diam}(\hat{K}) = O(1)$

$$x = F_K(\hat{x}) = \underbrace{B_K}_{d\times d\text{ — nonsingular matrix}} \hat{x} + \underbrace{\vec{b}_K}_{d\times 1\text{ — vector}}$$

Proposition $\quad \forall v \in H^m(K)$ with $m \geq 0$, define

$$\hat{v} = v \circ F_K \qquad \left(\hat{v}(\hat{x}) = v(F_K(\hat{x})) \right)$$

$\Longrightarrow \hat{v} \in H^m(\hat{K})$ and $\exists$ const $C(m,d)$ s.t.

(i) $\quad |\hat{v}|_{m,\hat{K}} \leq C \|B_K\|^m |\det B_K|^{-\frac{1}{2}} |v|_{m,K} \quad \forall v \in H^m(K)$

(ii) $\quad |v|_{m,K} \leq C \|B_K^{-1}\|^m |\det B_K|^{\frac{1}{2}} |\hat{v}|_{m,\hat{K}} \quad \forall \hat{v} \in H^m(\hat{K})$

where $\quad \|B_K\| = \sup_{0 \neq x \in \mathbb{R}^d} \dfrac{\|B_K x\|_{\ell^2}}{\|x\|_{\ell^2}}$.

Proof $C^\infty(K)$ is dense in $H^m(K)$, it suffices to prove (i) for $v \in C^\infty(K)$.

$m=1$

$$|\hat{v}|^2_{1,\hat{K}} = \int_{\hat{K}} \|\hat{\nabla}\hat{v}\|^2_{l_2}\, d\hat{x}$$

$$= \int_K \|B_K \nabla v\|^2_{l_2} |\det B_K^{-1}|\, dx$$

$$\leq \|B_K\| |\det B_K^{-1}| \int_K \|\nabla v\|^2_{l_2}\, dx$$

$$= \|B_K\| |\det B_K^{-1}| |v|^2_{1,K}$$

$$\left[\begin{array}{l} \hat{v}(\hat{x}) = v(F_K(\hat{x})) \\ \dfrac{\partial \hat{v}}{\partial \hat{x}_1} = \dfrac{\partial v}{\partial x_1}\cdot\dfrac{\partial x_1}{\partial \hat{x}_1} + \dfrac{\partial v}{\partial x_2}\cdot\dfrac{\partial x_2}{\partial \hat{x}_1} = \left(\dfrac{\partial x_1}{\partial \hat{x}_1}, \dfrac{\partial x_2}{\partial \hat{x}_1}\right)\begin{pmatrix} \dfrac{\partial v}{\partial x_1} \\ \dfrac{\partial v}{\partial x_2}\end{pmatrix} \\ \hat{\nabla}\hat{v} = \begin{pmatrix} \dfrac{\partial x_1}{\partial \hat{x}_1} & \dfrac{\partial x_2}{\partial \hat{x}_1} \\ \dfrac{\partial x_1}{\partial \hat{x}_2} & \dfrac{\partial x_2}{\partial \hat{x}_2} \end{pmatrix} \nabla v = B_K \nabla v \\ x = B_K \hat{x} + \vec{b}_K \Rightarrow B_K = \dfrac{\partial x}{\partial \hat{x}} \\ d\hat{x} = \left|\det\left(\dfrac{\partial \hat{x}}{\partial x}\right)\right| dx = |\det B_K^{-1}|\, dx \end{array}\right.$$

#

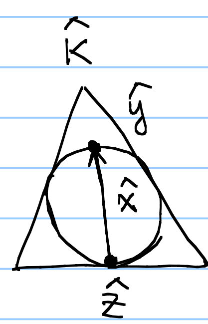
$h_K = \operatorname{diam}(K)$, $\hat{h} = \operatorname{diam}(\hat{K})$, $\rho_K = \sup\{\operatorname{diam}(S) \mid S \text{ is a ball contained in } K\}$

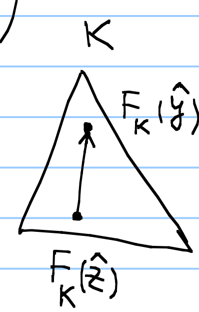
$\hat{\rho} = \sup\{\operatorname{diam}(S) \mid S \text{ is a ball contained in } K\}$.

Proposition Assume that K and $\hat{K}$ are affine-equivalent

$\left(\exists F_K : \hat{x} \in \mathbb{R}^n \longrightarrow x = F_K(\hat{x}) = B_K\hat{x} + \vec{b}_K \in \mathbb{R}^n \text{ s.t. } K = F_K(\hat{K})\right.$
$\left.\quad \text{where } B_K \text{ is nonsingular}\right)$

$$\Longrightarrow \|B_K\| \leq \frac{h_K}{\hat{\rho}} \quad \text{and} \quad \|B_K^{-1}\| \leq \frac{\hat{h}}{\rho}.$$





Proof $$\|B_K\| = \sup_{\hat{x}\in\mathbb{R}^n} \frac{\|B_K\hat{x}\|}{\|\hat{x}\|} = \sup_{\|\hat{x}\| = \hat{\rho}} \frac{\|B_K\hat{x}\|}{\hat{\rho}}$$

For $\hat{x}$ with $\|\hat{x}\| = \hat{\rho}$, $\exists\, \hat{y}, \hat{z} \in \overline{\hat{K}}$ s.t. $\hat{x} = \hat{y} - \hat{z}$

$$\Longrightarrow B_K\hat{x} = B_K\hat{y} - B_K\hat{z} = F_K(\hat{y}) - F_K(\hat{z})$$

$$\Longrightarrow \|B_K\hat{x}\| = \|F_K(\hat{y}) - F_K(\hat{z})\| \leq h_K \quad \left(F_K(\hat{y}), F_K(\hat{z}) \in \overline{K}\right)$$

#

Thrm For $0 \le m \le k+1$ with $k \ge 1$, $\exists$ const $C = C(\hat{K}, I^k_{\hat{K}}, k, m, d)$ s.t.

$$|v - I^k_K(v)|_{m,K} \le C \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,K} \quad \forall v \in H^{k+1}(K).$$

Proof I^k_K is well defined in $H^{k+1}(K)$ since $H^{k+1}(K) \hookrightarrow C^0(\bar{K})$.

$$|v - I^k_K(v)|_{m,K} \le C \|B_K^{-1}\|^m |\det B_K|^{\frac{1}{2}} |\widehat{v - I^k_K(v)}|_{m,\hat{K}}$$

$$\widehat{v - I^k_K(v)} = (v - I^k_K(v)) \circ F_K = \hat{v} - I^k_K(v) \circ F_K$$

$$= \hat{v} - \sum_i v(a_{i,K}) \varphi_i \circ F_K \qquad I^k_K(v) = \sum_i v(a_{i,K}) \varphi_i(x)$$

$$= \hat{v} - \sum_i v(F_K(\hat{a}_i)) \hat{\varphi}_i \qquad \hat{\varphi}_i = \varphi_i \circ F_K$$

$$= \hat{v} - \sum_i \hat{v}(\hat{a}_i) \hat{\varphi}_i \qquad \hat{v}(\hat{a}_i) = v(F_K(a_i))$$

$$= \hat{v} - I^k_{\hat{K}}(\hat{v})$$

$$\|B_K^{-1}\|^m \le C \frac{\hat{h}^m}{\rho_K^m} = C \rho_K^{-m}$$

$$\Rightarrow |v - I^k_K(v)|_{m,K} \le C \rho_K^{-m} |\det B_K|^{\frac{1}{2}} |\hat{v} - I^k_{\hat{K}}(\hat{v})|_{m,\hat{K}} \qquad (*)$$

$$\boxed{|\hat{v} - I^k_{\hat{K}} \hat{v}|_{m,\hat{K}}} = |(I - I^k_{\hat{K}})(\hat{v})|_{m,\hat{K}} = |(I - I^k_{\hat{K}})(\hat{v} + \hat{p})|_{m,\hat{K}} \quad \forall \hat{p} \in \mathcal{P}_k \qquad \left((I - I^k_{\hat{K}})\hat{p} = 0\right)$$

Bramble-Hilbert Lemma

$$\boxed{\le \|I - I^k_{\hat{K}}\|_{\mathcal{L}(H^{k+1}(\hat{K}), H^m(\hat{K}))} \inf_{\hat{p} \in \mathcal{P}_k} \|\hat{v} + \hat{p}\|_{k+1,\hat{K}}}$$

Deny-Lions Lemma

$$\le C(I^k_{\hat{K}}, \hat{K}) |\hat{v}|_{k+1,\hat{K}}$$

$$|\hat v|_{k+1,\hat K} \le C \|B_K\|^{k+1} |\det B_K|^{-\frac12} |v|_{k+1,K}$$
$$\le C h_K^{k+1} |\det B_K|^{-\frac12} |v|_{k+1,K}$$

$$\Longrightarrow \left|\hat v - I_{\hat K}^k(\hat v)\right|_{m,\hat K} \le C h_K^{k+1} |\det B_K|^{-\frac12} |v|_{k+1,K} \qquad (**)$$

$$(*) \text{ and } (**) \Longrightarrow \left|v - I_K^k(v)\right|_{m,K} \le C h_K^{k+1} \rho_K^{m} |v|_{k+1,K} \qquad \#$$

Remark

(1) $\forall\ v \in H^{\underline{\underline{l+1}}}(K)$ with $1 \le l < k$

$$|I - I_K^k(v)|_{m,K} \le C h_K^{l+1} \rho_K^{-m} |v|_{l+1,K} \quad \text{for } m = 0, 1, \cdots, l+1.$$

(2) similar results hold for interpolation in $W^{k+1,p}(K)$

(3)

$$|v - I_K^k v|_{m,\infty,K} \le C \begin{cases} |K|^{-\frac12} h_K^{l+1} \rho_K^{-m} |v|_{l+1,K} & 1 \le l \le k,\ 0 \le m < l+1-\frac{d}{2} \\ h_K^{l+1} \rho_K^{-m} |v|_{l+1,\infty,K} & 1 \le l \le k,\ 0 \le m \le l+1 \end{cases}$$

<u>Regular Triangulation</u> A family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is regular.

$$\Longleftrightarrow \max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \le \sigma \quad \forall\ h > 0.$$

<u>Thrm</u> Let $\mathcal{T}_h$ be regular and $m=0,1$, $k\geq 1$. $\forall\, v\in H^{l+1}(\Omega)$ with $1\leq l\leq k$

$$\Longrightarrow \quad \left|v - I_h^k(v)\right|_{m,\Omega} \leq C h^{l+1-m} |v|_{l+1,\Omega}$$

<u>Deny-Lions Lemma</u>

$\forall\, v\in H^{k+1}(K)$, $\exists$ const $C(k,K)$ s.t.

$$\inf_{p\in\mathcal{P}_k} \|v+p\|_{k+1,K} \leq C\,|v|_{k+1,K}.$$

<u>Proof</u>

First, prove that $\exists$ const $C=C(K)>0$ s.t.

$$\|v\|_{k+1,K} \leq C\left[|v|^2_{k+1,K} + \sum_{|\alpha|\leq k}\left(\int_K D^\alpha v\,dx\right)^2\right]^{\frac{1}{2}} \quad \forall\, v\in H^{k+1}(K) \qquad (*)$$

Now, for $v\in H^{k+1}(K)$, $\exists 1\ \xi\in P_k$ s.t. $\int_K D^\alpha \xi\,dx = -\int_K D^\alpha v\,dx \quad \forall\, |\alpha|\leq k$

$\dim \mathcal{P}_k = \#\left\{\alpha=(\alpha_1,\cdots,\alpha_d)\,\middle|\, \alpha_i - \text{integer},\ |\alpha|\leq k\right\}$

$$\Longrightarrow \inf_{p\in P_k} \|v+p\|_{k+1,K} \leq \|v+\xi\|_{k+1,K} \leq C\left[|v+\xi|^2_{k+1,K} + 0\right]^{\frac{1}{2}} = C|v|_{k+1,K} \quad \#$$

<u>Proof of $(*)$</u> (compactness argument) Assume that $(*)$ is wrong.

$$\Longrightarrow \exists \{v_j\}_{j=1}^\infty \subset H^{k+1}(K) \text{ s.t.}$$

$$\|v_j\|_{k+1,K} = 1 \quad \text{but} \quad |v_j|^2_{k+1,K} + \sum_{|\alpha|\leq k}\left(\int_K D^\alpha v_j\,dx\right)^2 < \frac{1}{j}$$

$X \hookrightarrow Y$: (1) $X \subset Y$; (2) $\|\cdot\|_Y \le C \|\cdot\|_X$
is compact
(3) any bounded set in X is precompact in Y.
(every bounded seq. in X has a subseq that is Cauchy in Y)

$$\left.\begin{array}{l} H^{k+1}(K) \hookrightarrow H^k(K) \text{ is compact} \\ \|v_j\|_{k+1,K} = 1 \end{array}\right\} \Rightarrow \exists \{v_{j_\ell}\} \text{ that converges in } H^k(K)$$

$$\Rightarrow \{v_{j_\ell}\} \text{ is a Cauchy sequence in } H^{k+1}(K)$$

$$\|v_{j_\ell} - v_{j_{\ell'}}\|^2_{k+1,K} = |v_{j_\ell} - v_{j_{\ell'}}|^2_{k+1,K} + \|v_{j_\ell} - v_{j_{\ell'}}\|^2_{k,K} \longrightarrow 0$$

$$\Rightarrow \exists\, w \in H^{k+1}(K) \text{ s.t. } v_{j_\ell} \to w \text{ in } H^{k+1}(K)$$

$$\Rightarrow \|w\|_{k+1,K} = \overline{\lim_{\ell\to\infty}} \|v_{j_\ell}\|_{k+1,K} = 1$$

But

$$0 = \overline{\lim_{\ell\to\infty}} \left[|v_{j_\ell}|^2_{k+1,K} + \sum_{|\alpha|\le k} \left(\int_K D^\alpha v_{j_\ell}\, dx \right)^2 \right] = |w|^2_{k+1,K} + \sum_{|\alpha|\le k} \left(\int_K D^\alpha w\, dx \right)^2$$

$$\Rightarrow \left.\begin{cases} D^\alpha w = 0 & |\alpha| = k+1 \implies w \in P_k \\ \int_K D^\alpha w\, dx = 0 & |\alpha| \le k \end{cases}\right\} \quad w \equiv 0$$

contradictory with $\|w\|_{k+1,K} = 1$

#

Inverse Estimates

Thrm Let $\mathcal{T}_h$ be regular. Then $\exists\ c>0$ s.t.

$$\|v\|_t \le c\, h^{m-t} \|v\|_m \quad \forall v \in S_h, \ \forall\ 0 \le m \le t.$$

Clément-Type Interpolation

standard interpolation $\quad I_h v\big|_K = I_K v$ and $I_K v = \sum \underline{v(a_i)}\, \varphi_i(x)$

Let $S_h^1 = \left\{ v \in C^0(\bar{\Omega}) \,\middle|\, v\big|_K \in P_1(K)\ \forall K \in \mathcal{T}_h \right\}$

$= \operatorname{span}\{ \varphi_i(x) \}$

$\forall v \in L^2(\Omega)$, define

$$I_h v = \sum_i v_i \varphi_i(x) \quad \text{with} \quad v_i = \frac{1}{|\omega_i|}\int_{\omega_i} v\,dx, \quad \omega_i = \operatorname{supp}\{\varphi_i\}$$

§7. Errors Bounds for Elliptic Problems of 2^{nd} Order

Remarks on Regularity

$$\begin{cases} -\operatorname{div}(A\nabla u) + a_0 u = f \text{ in } \Omega \\ u\big|_{\Gamma_D} = g_D \text{ and } \vec{n}\cdot A\nabla u\big|_{\Gamma_N} = g_N \end{cases} \qquad \begin{cases} \text{Find } u \in H^1_{g,D}(\Omega) \text{ s.t.} \\ a(u,v) = f(v) \quad \forall v \in H^1_{0,D}(\Omega) \end{cases}$$

It is H^s-regular $(s \geq 1)$ if

$$\|u\|_s \leq c\left(\|f\|_{s-2} + \|g_D\|_{s-\frac{1}{2},\Gamma_D} + \|g_N\|_{s-\frac{3}{2},\Gamma_N}\right)$$

Let $g_D = 0$ and $g_N = 0$.

Error Bounds in the Energy Norm

Let $\mathcal{T}_h$ be a regular triangulation, $S_h^k = \left\{ v \in C^0(\bar{\Omega}) \,\middle|\, v\big|_K \in P_k(K)\ \forall K \in \mathcal{T}_h \right\}$.

$u_h \in S_h^k$ be the FEA, then

$$\|u - u_h\|_1 \leq ch\|u\|_2 \leq ch\|f\|_0$$

Remark 7.4.
- $O(h^2)$ is not possible due to non-smooth $\partial\Omega$
- quadratic & cubic are better than linear even if $u \notin H^3(\Omega)$.

L^2-Estimates If $u \in H^2(\Omega)$, then

$$\|u - u_h\| \leq ch^2\|f\|_0 .$$

L^∞-Estimates

$$\|u - u_h\|_\infty \leq ch^2 |\log h|^{\frac{3}{2}} \|D^2 u\|_\infty .$$

§8 Computational Considerations

- construction of a grid by partitioning Ω
  setting up the stiffness matrix
- solution of the system of algebraic equations

Adaptive Mesh Refinement

Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine.